

Situation: Increasing or Decreasing

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Prompt

In a math modeling class a student is working through a test review and comes across the following question, “Find where the function either increases or decreases and where the rate of change either increases or decreases.” The student asks the teacher, “Won’t the intervals be the same for both? What is the difference between where the function is increasing and decreasing and where the rate of change is increasing and decreasing?”

Commentary

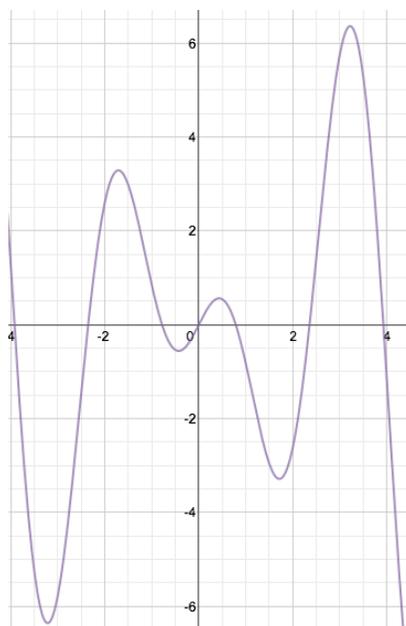
This situation stems from a Math 1101- Math Modeling class. While taking a pre-test five to six students were confused by the problem posed above. The teacher took roughly twenty minutes re-explaining the differences. There is a significant difference between the increasing and decreasing intervals of a function and the increasing and decreasing intervals of rate of change. The following foci are designed to better prepare teachers to make instructional decisions in order to best explain the prompt above no matter what level of mathematics is being taught. However, do note that the following foci will only address strictly increasing and decreasing differentiable functions.

Mathematical Foci

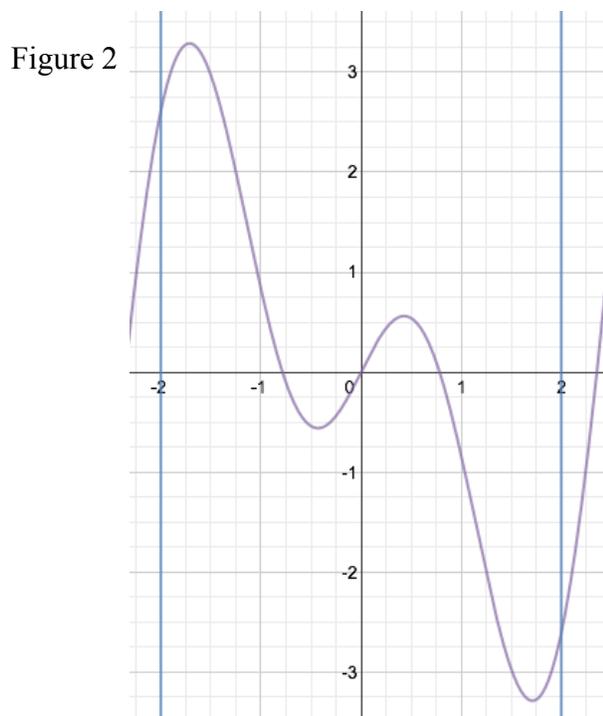
Focus 1: *Strictly Increasing and Decreasing Differentiable Functions*

A function is defined as increasing if the y-values are increasing as the x-values are increasing. In general, for any coordinate points within the interval (x_1, y_1) and (x_2, y_2) , if $x_1 < x_2$ then $y_1 < y_2$ and the function will be increasing through the interval.

Figure 1



Looking at Figure 1 above we can see there are several intervals where the function is increasing. Now looking below at Figure 2 and using the definition above find where the function increasing over the interval $[-2, 2]$. The lines $x = -2$ and $x = 2$ have been drawn in to help see the interval of interest.



Again, using the definition we can roughly estimate that the function is increasing over the intervals $[-2, -1.75)$, $(-0.4, 0.4)$, and $(1.75, 2]$. In order to find these intervals we had to rely on estimation since the grid was not precise. Decreasing and increasing intervals are written in reference to the x-values. Therefore, in the past example the intervals represent x-values only.

Notice above there is a mixture of brackets and parenthesis in the set of increasing intervals. The use of brackets and parentheses are necessary in order to specify which values are included or not included in the interval. Those numbers adjacent to a bracket are considered included in the interval; whereas the numbers adjacent to a parenthesis are considered not included in the interval. (For more information on when to use brackets or parentheses see Foy's Situation 3 (Foy, 2013, p.1).) In this particular example, the values -1.75 , -0.4 , 0.4 , and 1.75 are not included in the interval of increasing x-values because they are located at either a maximum or minimum point of the function.

Maxima and minima points are tricky for students for they are neither increasing nor decreasing. This is one reason for why we cannot label the point one way or the other. Maxima and minima are found by taking the derivative of the function and setting it equal to zero in order to solve for x. Where the derivative equals zero there you will find a maximum or minimum point. The topic of derivatives and differentiability of functions will be discussed further in Focus 3.

A function is defined as decreasing if the y-values are decreasing as the x-values are increasing. In general, for any coordinate points within the interval (x_1, y_1) and (x_2, y_2) , if $x_1 < x_2$ then $y_1 > y_2$ and the function will be decreasing through the interval. Again, look at Figure 2 and use the definition to find where the function is decreasing over the interval $[-2, 2]$. The function is decreasing over the intervals $(-1.75, -0.4)$ and $(0.4, 1.75)$. Notice if the intervals of increase and decrease are put together we get the entire interval $[-2, 2]$ as desired.

$$[-2, -1.75), (-1.75, -0.4), (-0.4, 0.4), (0.4, 1.75) \text{ and } (1.75, 2]$$

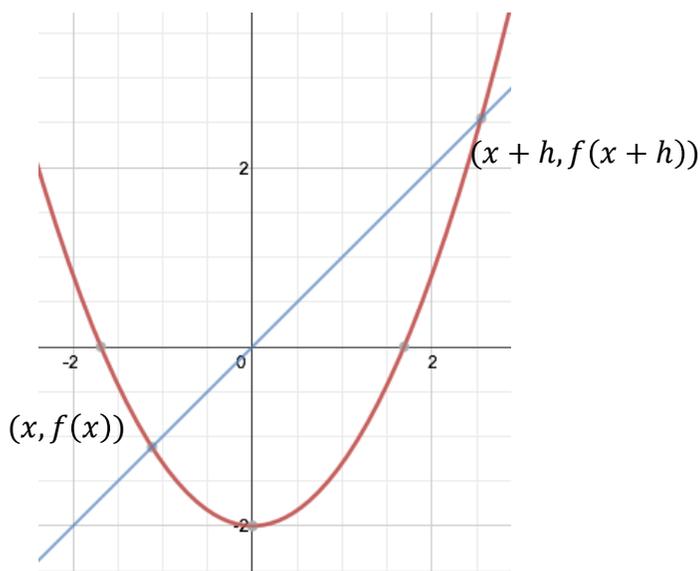
*The navy blue color indicates intervals of decrease. The dark red indicates the intervals of increase.

In conclusion, when determining where the differentiable function is strictly increasing and decreasing one must look at the y-values and see how they change as the x-values increase. When the y-values are increasing as the x-values are increasing the function will also be increasing, but when the y-values are decreasing as the x-values are increasing the function will be decreasing.

Focus 2: Average Rate of Change

The average rate of change of a function is the average rate at which the output value changes with respect to the input value. In order to calculate the average rate of change of a function the following definition may be used: $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$

$$f(x) = .7x^2 - 2$$



This definition should look familiar because it is simply the formula used to calculate slope. Finding the average rate of change means finding the slope of the line between two points. Above the function, $f(x) = .7x^2 - 2$, is intersected by the line $y = x$. The two points of intersection are shown above as $(x, f(x))$ and $(x+h, f(x+h))$. Using these two points to find

the slope, i.e. the change in y over the change in x we end up with the definition of the rate of change: $\frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{h}$.

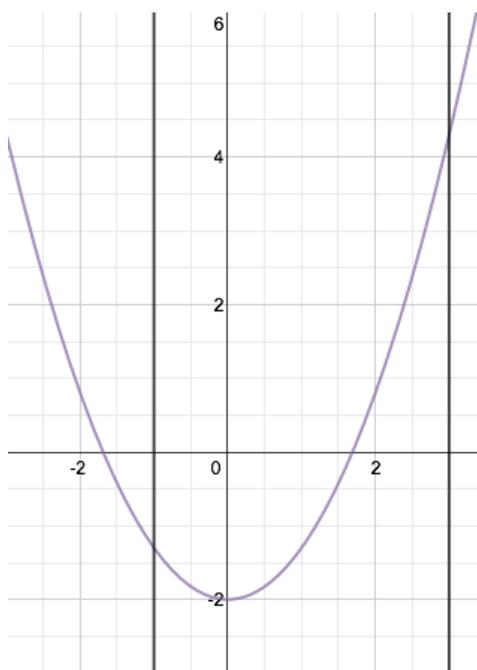
Recall from Geometry that a line that intersects a circle at two points is called a secant line. We will also call a line that intersects a function at two points a secant line.

A second way to define average rate of change is to use the following definition:

$$\frac{f(b) - f(a)}{b - a}$$

The definition above represents the rate of change over some interval from $t = a$ to $t = b$ of the function $f(t)$. This definition is similar to the first definition, but instead of using specific points the definition uses an interval. Looking at the graph below we have the same function as above, but restricted over the interval $[-1, 3]$.

$$f(x) = .7x^2 - 2$$



In order to find the rate of change of over the interval $[-1, 3]$ we must solve the following:

$$\frac{f(3) - f(-1)}{3 - (-1)} = \frac{(.7(3^2) - 2) - (.7(-1^2) - 2)}{4} = \frac{4.3 + 1.3}{4} = \frac{5.6}{4} = 1.4$$

Using both the graph and the calculations above we find that the slope of the secant line, i.e. the line intersecting the graph f at the points $(a, f(a))$ and $(b, f(b))$, is $\frac{\Delta y}{\Delta x} = \frac{5.6}{4}$.

In conclusion, the average rate of change of a function f is the slope of the secant line. The secant line is a line that intersects the function at two distinct points. The slope of this line can

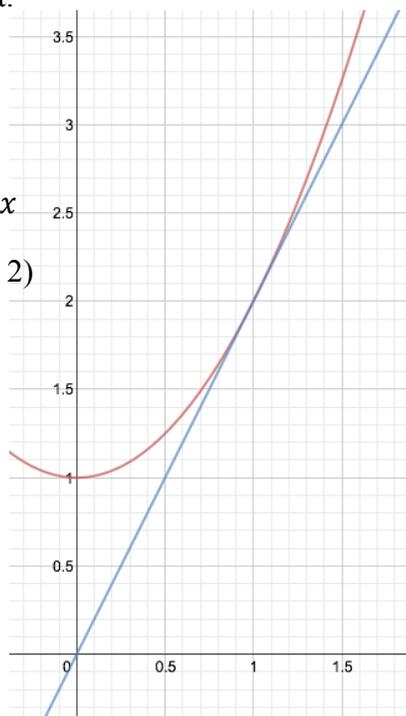
be found using the definition $\frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{h}$. The rate of change can also be found over a specific interval using the following definition: $\frac{f(b)-f(a)}{b-a}$. The average rate of change is an important concept when dealing with non-constant functions. Through this concept we are able to calculate and observe how the independent variable changes as the dependent variable changes.

Focus 3: Instantaneous Rate of Change

Understanding the concept of *average* rate of change is a stepping stone for students and teachers to understand *instantaneous* rate of change. It is important to note that the average rate of change and instantaneous rate of change cannot be used interchangeably. The differences will be made apparent and discussed in this focus.

Instantaneous rate of change is the rate of change at one specific moment. This focus will discuss two ways of finding the instantaneous rate of change. The first way that will be discussed is finding the slope of the tangent line at the specific point. A tangent line is defined as a straight or linear line that touches a curve at a single point.

Curve: $y = x^2 + 1$
 Tangent Line: $y = 2x$
 Point of tangency: $(1, 2)$



The line $y = 2x$ touches the curve $y = x^2 + 1$ at exactly one and only one point $(1, 2)$, which is called the point of tangency. Because the instantaneous rate of change at a specific point is the $\frac{\Delta y}{\Delta x}$ of the tangent line at that specific point, we can say the instantaneous rate of change is the slope of the tangent line, i.e. $m = \frac{2}{1}$. Reiterating, the instantaneous rate of change is 2.

The previous example shows a specific curve with a specific tangent line. Therefore, when both equations are given finding the slope of the tangent line is not difficult. However, if the equations are not given and more specifically the equation of the tangent line then a method of

estimation could be used or the method of differentiation in order to find the instantaneous rate of change. The former requires finding the slope of the secant line between two points very very close to the point of tangency. This will only give an estimation of the instantaneous rate of change. The latter method entails a method that will be discussed in the following paragraph.

The second method of finding the instantaneous rate of change is through differentiation. Finding the slope of secant lines between two points as close as possible to the desired point of tangency is called taking the $Limit_{h \rightarrow 0}$, the limit as h approaches zero. $Limit_{h \rightarrow 0}$ means the distance h between the x -values is getting smaller and smaller, i.e. approaching zero. Thus, the finding the *limit* means finding the instantaneous rate of change at the specific point. The following describes the process of finding the limit at a specific point:

$$f'(x) = \frac{dx}{dy} = Limit_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = Limit_{h \rightarrow 0} \frac{\Delta y}{\Delta x}$$

$\frac{dx}{dy}$ is the notation for the slope of the tangent line and $\frac{\Delta y}{\Delta x}$ is the slope of the secant line.

Hopefully, the definitions above looks familiar if not refer back to Focus 2.

Evaluating the instantaneous rate of change at a specific point x of a strictly increasing and decreasing differentiable function is found in two more ways. The first is by using the definition $Limit_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ or taking the derivative $\frac{dx}{dy}$. These two ways are the same and will produce the same answer, but with each uses different steps. Example of the first method:

Find the derivative using the definition of a limit: $f(x) = 2x^2 - 16x + 35$

$$\begin{aligned} F'(x) &= Limit_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= Limit_{h \rightarrow 0} \frac{2(x+h)^2 - 16(x+h) + 35 - (2x^2 - 16x + 35)}{h} \\ &= Limit_{h \rightarrow 0} \frac{2x^2 + 4xh + 2h^2 - 16x - 16h + 35 - 2x^2 + 16x - 35}{h} \\ &= Limit_{h \rightarrow 0} \frac{4xh + 2h^2 - 16h}{h} \\ &= Limit_{h \rightarrow 0} 4x + 2h - 16 \\ &= 4x + 2(0) - 16 \\ f'(x) &= 4x - 16 \end{aligned}$$

Therefore, anywhere on the function the instantaneous rate of change for any value of x will be evaluated using $f'(x) = 4x - 16$.

Now, finding the instantaneous rate of change using derivatives comes with some rules. According to Paul's Notes, "a function $f(x)$ is differentiable at $x = a$ if $f'(x)$ exists and $f(x)$ is called differentiable on an interval if the derivative exists for each point in that interval" (Dawkins, 2003, p.1). Again, evaluating the derivative and evaluating the limit are the same processes, but with different steps. Please see the following link for a list of common derivative rules as a refresher: http://www.mathwords.com/d/derivative_rules.htm.

Example: Using the rules of differentiation find the derivative of the function:

$$f(x) = 2x^2 - 16x + 35$$

$$f'(x) = 4x - 16$$

Does this evaluation answer look similar to the one previously explored? Yes it does! How many steps were taken to find this derivative? Two. Now, both ways are perfectly fine to use, but using the rules of derivatives is quicker and requires less algebra. However, note that both methods brought about the same derivative, which can be evaluated at any point x on the strictly differentiable function.

History:

The tangent line was first defined as, "a right line which touches a curve, but which when produced, does not cut it." However, this definition does not leave room for points of inflection to have a tangent. Gottfried Wilhelm von Leibniz, Sir Isaac Newton, and Pierre de Fermat later developed the modern definition of a tangent line around the time of 1680. The men pulled information, theories, and insights from each other to formulate the correct definition for the tangent line.

Post Commentary

Overall, the methods discussed above for finding the average rate of change and instantaneous rate of change are important to know and understand. There are many options to take when answering a student's question such as the questions posed in the prompt. Hopefully, this situation clarifies misconceptions and brings forth ways to teach where the function either increases or decreases and where the rate of change either increases or decreases

References

- Dawkins, P. (n.d.). Pauls Online Notes : Calculus I - The Definition of the Derivative. *Pauls Online Math Notes*. Retrieved July 31, 2013, from <http://tutorial.math.lamar.edu/Classes/Calcl/DefnOfDerivative.aspx>
- Foy, Colleen. "EMAT 6500 Connections in Secondary Mathematics." *Jim Wilson's Home Page*. N.p., n.d. Web. 24 July 2013. <<http://jwilson.coe.uga.edu/EMAT6500/EMAT6500.html>>.
- "Introduction: Average Rate of Change." *WindStream*. N.p., n.d. Web. 25 July 2013. <home.windstream.net/okrebs/page201.html>.
- Schaufele, Christopher. "Average Rate of Change." *Earth Math*. N.p., n.d. Web. 25 July 2013. <earthmath.kennesaw.edu/main_site/review_topics/rate_of_change.htm>.
- Simmons, B. (n.d.). Mathwords: Derivative Rules. *Mathwords*. Retrieved July 31, 2013, from http://www.mathwords.com/d/derivative_rules.htm
- Noah Webster, *American Dictionary of the English Language* (New York: S. Converse, 1828), vol.2, p.733
- "Tangent - Wikipedia, the free encyclopedia." *Wikipedia, the free encyclopedia*. N.p., n.d. Web. 25 July 2013. <<http://en.wikipedia.org/wiki/Tangent>>.